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The response of a laminar boundary layer in supersonic flow to small-amplitude progressive waves

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In this paper the effect of a small-amplitude progressive wave on the laminar boundary layer on a semi-infinite flat plate, due to a uniform supersonic free-stream flow is considered. The perturbation to the flow caused by the wave divides into two streamwise zones. In the first, relatively close to the leading edge of the plate, on a transverse scale comparable with the boundary-layer thickness, the perturbation flow is described by a form of the unsteady linearized compressible boundary-layer equations. In the free stream, this component of flow is governed by the wave equation, the solution of which provides the outer velocity conditions for the boundary layer. This boundary-layer system is solved numerically, and also the asymptotic structure in the far downstream limit is studied. This reveals a breakdown and a subsequent second streamwise zone, where the flow disturbance is predominantly inviscid. The two zones are shown to match in a proper asymptotic sense.

1. Introduction

Much work has been carried out on the response of a two-dimensional incompressible laminar boundary layer on a semi-infinite flat plate to time-harmonic oscillatory perturbations (particularly those of infinite wavelength in the streamwise direction) of the free stream. Most of the effort has been involved with small-amplitude perturbations; included in this category are the works of Lighthill (1954), Lam & Rott (1960) and Ackerberg & Phillips (1972). Close to the leading edge, the flow is of Blasius type, whilst far downstream the boundary layer takes on a double structure, comprising an inner Stokes-type layer, and an outer Blasius-type layer.

Numerical investigations of this small-amplitude problem, describing the flow from the leading edge to far downstream, have been conducted by Ackerberg & Phillips (1970) and Goldstein, Sockol & Sanz (1983). Some details of the far-downstream behaviour of this problem have in the past been the subject of some controversy. At large distances downstream of the leading edge, a set of eigensolutions must exist, which decay exponentially downstream (this reflects the effect of the particular conditions prevailing upstream). One set of eigensolutions found by Lam & Rott (1960) and by Ackerberg & Phillips (1972) is determined by conditions close to the wall, and has decay rates that *decrease* as the order increases (and so the most important modes are those of highest order). A second set of eigensolutions has been found by Brown & Stewartson (1973*a*, *b*), and these are determined primarily by conditions at the outer edge of the boundary layer, and are characterized by a decay rate that *increases* with increasing order. Some discussion of the relationship between these two sets of modes is to be found in Goldstein *et al.* (1983).

Further downstream still, Goldstein (1983) showed how the eigensolutions of Lam & Rott (1960), which initially decay, develop into classical, high Reynolds-number-limit Tollmien–Schlichting modes (including unstable modes) for Blasius-type boundary layers. Although the Lam & Rott (1960) eigensolutions decay downstream, they also oscillate with increasingly rapid (spatial) frequency, and far downstream, transverse and streamwise gradients must become comparable, giving rise to an entirely new structure in which transverse pressure gradients play a key role.

Returning to the boundary-layer problem, non-small oscillations of the free stream (but still limited to non-reversing free streams) have been tackled by Moore (1951, 1957) and Pedley (1972) for regions close to the leading edge of the plate, and it is again found that the flow is Blasius like. Lin (1956), Gibson (1957) and Pedley (1972) studied the far downstream region, and found that the flow takes on a double structure similar to that observed in the analogous small-amplitude case. Duck (1989) has presented numerical results which extend from the leading edge to far downstream, in this case.

All these aforementioned papers were concerned with purely temporal flow oscillations (i.e. oscillations of infinite streamwise wavelength). The effect of a small-amplitude progressive wave on an incompressible boundary layer has been investigated by Kestin, Maeder & Wang (1961) and Patel (1975). The former authors considered the low-frequency limit to the problem, for the very particular case when the wave speed equalled the mean far-field velocity. Patel (1975) gave results for both high and low frequency, the latter being obtained using an empirical momentum integral approach. A number of experimental results were also presented, and compared with theory.

In this paper we consider the effect of a small-amplitude two-dimensional progressive wave in a supersonic free stream, on a laminar boundary layer on a flat plate. Here the situation is rather different to the incompressible case in a number of respects. Although eigenfunctions similar to those of Lam & Rott (1960) undoubtedly exist, these are not expected to develop into growing Tollmien–Schlichting waves (at least not in the case of two-dimensional flows – see Ryzhov & Zhuk 1980 and Duck 1985). As a result, although planar Tollmien–Schlichting (i.e. viscous) waves are present downstream, these are expected to be of little consequence. The dominant planar modes of instability are likely to be inviscid in nature; however Smith (1989) and Duck (1990) have shown, using triple-deck theory, that oblique viscous modes do exist in supersonic boundary layers (confirming the numerical work of, for example, Mack 1984). Indeed, it is found that viscous modes can dominate over inviscid modes at low supersonic Mach numbers, at all but extremely large Reynolds numbers. In this paper, as well as presenting results for the boundary-layer region, we go on to consider the development of the unsteady component of flow far downstream, which turns out to be predominantly inviscid.

A related study involving the forced response of a boundary layer (at finite Reynolds number), disturbed by an oblique, moving wavy wall has been considered by Mack (1975); additional results for this problem have been given by Mack (1984).

The layout of the paper is as follows: in §2 we consider the 'fore region' in which the flow close to the plate is governed by a form of the compressible boundary-layer equations. We look at the downstream limit of this zone, and then in §3 we consider the downstream region, wherein transverse variations of pressure become important;

the regions studied separately in §§2 and 3 are seen in §4 to match (in an asymptotic sense). Finally in §5 we present a number of conclusions.

2. The fore region

We consider the viscous supersonic two-dimensional flow over a semi-infinite, stationary plate. We take the origin at the leading edge of the plate, x the coordinate along the plate, y the coordinate normal to the plate, and u and v the velocity components in the x- and y-directions respectively. t denotes time, ρ the fluid density and μ the first coefficient of viscosity of the fluid. p and T are defined to be the pressure and temperature in the fluid; $c_p(c_v)$ is the specific heat of the fluid at constant pressure (volume), and $\sigma = \mu c_p/K$ is the Prandtl number (K being the coefficient of thermal diffusivity); c_p, c_v and σ are all assumed to be constants. Subscript ∞ will be used to denote far-field, unperturbed conditions.

The fluid is also assumed to satisfy the following equation of state:

$$p = \rho RT, \tag{2.1}$$

where

$$R = c_p - c_v. \tag{2.2}$$

We consider first the fluid in the far field, where the effects of viscosity are expected to be negligible. In this region, we take the flow to comprise a uniform steady stream parallel to the plate, perturbed by a small-amplitude ($O(\epsilon) \ll 1$) progressive wave.

We therefore write

$$\begin{aligned} u &= U_\infty + \epsilon u_1(x,y,t) + \ldots, \quad v = \epsilon v_1(x,y,t) + \ldots, \\ \rho &= \rho_\infty + \epsilon \rho_1(x,y,t) + \ldots, \quad T = T_\infty + \epsilon T_1(x,y,t) + \ldots, \\ p &= p_\infty + \epsilon p_1(x,y,t). \end{aligned} \tag{2.3}$$

The leading-order equation of state gives

$$p_\infty = \rho_\infty R T_\infty. \tag{2.4}$$

Substitution of (2.3) into the momentum equations and continuity equation yields (at $O(\epsilon)$)

$$\rho_\infty \left(\frac{\partial u_1}{\partial t} + U_\infty \frac{\partial u_1}{\partial x} \right) = -\frac{\partial p_1}{\partial x}, \tag{2.5}$$

$$\rho_\infty \left(\frac{\partial v_1}{\partial t} + U_\infty \frac{\partial v_1}{\partial x} \right) = -\frac{\partial p_1}{\partial y}, \tag{2.6}$$

$$\frac{\partial \rho_1}{\partial t} + U_\infty \frac{\partial \rho_1}{\partial x} + \rho_\infty \frac{\partial u_1}{\partial x} + \rho_\infty \frac{\partial u_1}{\partial y} = 0. \tag{2.7}$$

If we write

$$a_\infty^2 = \gamma R T_\infty, \tag{2.8}$$

with

$$\gamma = c_p/c_v \quad \text{(assumed constant)}, \tag{2.9}$$

where a_∞ denotes the speed of sound in the far field, then from (2.5)–(2.7) we have

$$a_\infty^2 \left[\frac{\partial^2 p_1}{\partial x^2} + \frac{\partial^2 p_1}{\partial y^2} \right] = \left[\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} \right]^2 p_1. \tag{2.10}$$

We now seek progressive-wave solutions of this equation, writing

$$p_1 = \hat{p}_1\{\exp[\mathrm{i}\alpha(x-ct+\lambda y)]+\exp[\mathrm{i}\alpha(x-ct-\lambda y)]\} \tag{2.11}$$

(corresponding to an oblique wave, in general), there $\hat{p}_1$ is a constant. Substitution of (2.11) into (2.10) reveals that we must have

$$c = U_\infty\left[1\pm\frac{1}{M_\infty}(1+\lambda^2)^{\frac{1}{2}}\right], \tag{2.12}$$

where M_∞ denotes the unperturbed free stream Mach number, U_∞/a_∞.

It is now possible to write the progressive-wave solution for u_1 and v_1, namely

$$u_1 = -\frac{\hat{p}_1}{(U_\infty-c)\,\rho_\infty}\{\exp[\mathrm{i}\alpha(x-ct+\lambda y)]+\exp[\mathrm{i}\alpha(x-ct-\lambda y)]\}, \tag{2.13}$$

$$v_1 = -\frac{\lambda\hat{p}_1}{(U_\infty-c)\,\rho_\infty}\{\exp[\mathrm{i}\alpha(x-ct+\lambda y)]-\exp[\mathrm{i}\alpha(x-ct-\lambda y)]\}. \tag{2.14}$$

Equation (2.14) also satisfies the normal flow condition, although the no-slip condition is violated by (2.13), since

$$u_1(y=0) = -\frac{2\hat{p}_1}{(U_\infty-c)\,\rho_\infty}\exp[\mathrm{i}\alpha(x-ct)]. \tag{2.15}$$

Notice in particular that in the case of plane waves ($\lambda = 0$),

$$c = U_\infty\left[1\pm\frac{1}{M_\infty}\right], \tag{2.16}$$

$$v_1 = 0 \tag{2.17}$$

(whilst (2.15) is unaltered).

The slip velocity is reduced to zero, in the usual way, by the inclusion of a thin boundary layer. The boundary-layer approximation reduces the governing equations to (Stewartson 1964)

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{2.18}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0, \tag{2.19}$$

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-\frac{\partial p}{\partial t}-u\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu\frac{c_p}{\sigma}\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{2.20}$$

$$\frac{\partial p}{\partial y} = 0 \tag{2.21}$$

(together with (2.1)).

These equations can be simplified by the use of a generalization of the Howarth–Dorodnitsyn transformation, suggested by Stewartson (1951) and Moore (1951). We write

$$\rho = \rho_\infty\frac{\partial \bar{Y}}{\partial y}, \quad u = \frac{\partial \psi}{\partial \bar{Y}}, \tag{2.22}$$

where $\bar{Y}$ now replaces y as the independent transverse variable, and ψ is essentially a stream function. Then

$$v = -\frac{\rho_\infty}{\rho}\left[\left(\frac{\partial\psi}{\partial x}\right)_{\bar{Y},t} + u\left(\frac{\partial\bar{Y}}{\partial x}\right)_{y,t} + \left(\frac{\partial\bar{Y}}{\partial t}\right)_{x,y}\right], \tag{2.23}$$

and the equations of motion and energy reduce to

$$\frac{\partial^2\psi}{\partial\bar{Y}\partial t} + \frac{\partial\psi}{\partial\bar{Y}}\frac{\partial^2\psi}{\partial x\partial\bar{Y}} - \frac{\partial^2\psi}{\partial\bar{Y}^2}\frac{\partial\psi}{\partial x} = -\frac{T}{T_1\rho_1}\frac{\partial p_1}{\partial x} + \frac{\mu_\infty p_1}{p_\infty\rho_\infty}\frac{\partial}{\partial\bar{Y}}\left[\frac{\mu T_\infty}{\mu_\infty T}\frac{\partial^2\psi}{\partial\bar{Y}^2}\right], \tag{2.24}$$

$$\frac{\partial T}{\partial t} + \frac{\partial\psi}{\partial\bar{Y}}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial\bar{Y}} - \frac{T}{c_p\rho_1 T_1}\left(\frac{\partial p_1}{\partial t} + \frac{\partial\psi}{\partial\bar{Y}}\frac{\partial p_1}{\partial x}\right)$$
$$= \frac{\mu_\infty p_1}{\rho_\infty p_\infty}\frac{\partial}{\partial\bar{Y}}\left(\frac{\mu T_\infty}{\sigma\mu_\infty}\frac{\partial T}{T\partial\bar{Y}}\right) + \frac{p_1\mu T_\infty}{c_p\rho_\infty p_\infty T}\left(\frac{\partial^2\psi}{\partial\bar{Y}^2}\right). \tag{2.25}$$

On account of (2.21), we have written

$$p(x,\bar{Y},t) = p_1(x,t). \tag{2.26}$$

The usage of the stream function ensures that the continuity condition (2.19) is satisfied.

We now seek a solution to (2.24) and (2.25), subject to the boundary conditions

$$\frac{\partial T}{\partial\bar{Y}}(\bar{Y}=0) = 0, \tag{2.27}$$

$$T(\bar{Y}\to\infty)\to 1+O(\epsilon), \tag{2.28}$$

$$\psi(\bar{Y}=0) = \psi_{\bar{Y}}(\bar{Y}=0) = 0, \tag{2.29}$$

$$\psi_{\bar{Y}} \to U_\infty - \frac{2\hat{p}_1\epsilon}{(U_\infty-c)\rho_\infty}\exp\left[\mathrm{i}\alpha(x-ct)\right] \quad \text{as} \quad \bar{Y}\to\infty. \tag{2.30}$$

At this stage we must specify the particular viscosity/temperature model, and here we choose the simplest example, namely the linear law of Chapman (see Stewartson 1964),

$$\mu = \frac{C\mu_\infty T}{T_\infty}, \tag{2.31}$$

where C is the Chapman constant.

We now seek a perturbation solution (in powers of ϵ); we write

$$\begin{aligned} &\psi = \psi_0 + \epsilon\tilde{\psi} + \ldots, \quad T = T_0 + \epsilon\tilde{T} + \ldots, \\ &\rho = \rho_0 + \epsilon\tilde{\rho} + \ldots, \quad \bar{Y} = Y_0 + \epsilon\tilde{Y} + \ldots, \\ &p_1 = \rho_\infty RT_\infty + \epsilon\tilde{p} + \ldots. \end{aligned} \tag{2.32}$$

Substituting (2.32) into (2.24) and (2.25) yields, to $O(\epsilon^0)$,

$$\frac{\partial\psi_0}{\partial Y_0}\frac{\partial^2\psi_0}{\partial x\partial Y_0} - \frac{\partial\psi_0}{\partial x}\frac{\partial^2\psi_0}{\partial Y_0^2} = \frac{C\mu_\infty}{\rho_\infty}\frac{\partial^3\psi_0}{\partial Y_0^3}, \tag{2.33}$$

$$\frac{\partial\psi_0}{\partial Y_0}\frac{\partial T_0}{\partial x} - \frac{\partial\psi_0}{\partial x}\frac{\partial T_0}{\partial Y_0} = \frac{C\mu_\infty}{\sigma\rho_\infty}\frac{\partial^2 T_0}{\partial Y_0^2} + \frac{C\mu_\infty}{c_p\rho_\infty}\left(\frac{\partial^2\psi_0}{\partial Y_0^2}\right)^2. \tag{2.34}$$

The solution of ψ_0 (and T_0) is now routine, and merely corresponds to the similarity form of Blasius. Writing

$$\psi_0 = \left(\frac{2\mu_\infty U_\infty xC}{\rho_\infty}\right)^{\frac{1}{2}} F_0(\eta), \tag{2.35}$$

$$T_0 = T_\infty G_0(\eta), \tag{2.36}$$

where

$$\eta = Y_0\left(\frac{\rho_\infty U_\infty}{2\mu_\infty xC}\right)^{\frac{1}{2}} = \frac{Y_0}{\delta}, \tag{2.37}$$

and δ is a measure of the boundary-layer thickness, then

$$F_0''' + F_0 F_0'' = 0, \tag{2.38}$$

$$\frac{1}{\sigma}G_0'' + F_0 G_0' + (\gamma-1)M_\infty^2 F_0''^2 = 0. \tag{2.39}$$

The boundary conditions to be applied are

$$\begin{aligned} &F_0(0) = F_0'(0) = 0, \quad F_0(\infty) = 1, \\ &G_0'(0) = 0, \quad G_0(\infty) = 1. \end{aligned} \tag{2.40}$$

The $O(\epsilon)$ equations are rather more complicated, although their derivation is straightforward. The momentum and energy equations at this order are, respectively,

$$\tilde{\psi}_{Y_0 t} + \psi_{0Y_0}\tilde{\psi}_{xY_0} + \psi_{0xY_0}\tilde{\psi}_{Y_0} - \psi_{0x}\tilde{\psi}_{Y_0Y_0} - \tilde{\psi}_x\psi_{0Y_0Y_0} = -\frac{T_0}{T_\infty \rho_\infty}\frac{\partial \tilde{p}}{\partial x} + \frac{C\mu_\infty}{\rho_\infty}\frac{\partial^3\tilde{\psi}}{\partial Y_0^3} + \frac{C\tilde{p}\,\mu_\infty}{p_\infty\rho_\infty}\frac{\partial^3\psi_0}{\partial Y_0^3}, \tag{2.41}$$

$$\tilde{T}_t + \psi_{0Y_0}\tilde{T}_x - \psi_{0x}\tilde{T}_{Y_0} + T_{0x}\tilde{\psi}_{Y_0} - T_{0Y_0}\tilde{\psi}_x - \frac{T_0}{c_p\rho_\infty T_\infty}\left(\frac{\partial \tilde{p}}{\partial t} + \psi_{Y_0}\frac{\partial \tilde{p}}{\partial x}\right) = \frac{\mu_\infty C}{\rho_\infty \sigma}\tilde{T}_{Y_0Y_0} + \frac{\tilde{p}C\mu_\infty}{p_\infty \sigma}T_{0Y_0Y_0}. \tag{2.42}$$

Our primary concern here will be with the momentum equation (2.41) (although a similar treatment may be carried out on (2.42)).

We already have (2.35)–(2.37), and now we write

$$\tilde{\psi} = -\frac{2\hat{p}_1}{(U_\infty - c)\rho_\infty}\left(\frac{2\mu_\infty xC}{\rho_\infty U_\infty}\right)^{\frac{1}{2}} F_1(\eta,\xi)\,\mathrm{e}^{-\mathrm{i}\alpha ct}, \tag{2.43}$$

where

$$\xi = x^{\frac{1}{2}}. \tag{2.44}$$

These transformations, when applied to (2.41) yield the following equation for $F_1(\eta,\xi)$:

$$F_{1\eta\eta\eta} + F_0F_{1\eta\eta} + F_1F_{0\eta\eta} + \frac{2\mathrm{i}\alpha c}{U_\infty}\xi^2F_{1\eta} - \xi F_{0\eta}F_{1\eta\xi} + \xi F_{1\xi}F_{0\eta\eta} = 2G_0(\eta)\,\xi^2\mathrm{i}\alpha\left(\frac{c}{U_\infty}-1\right)\mathrm{e}^{\mathrm{i}\xi^2\alpha} - \gamma M_\infty^2\left(\frac{c}{U_\infty}-1\right)\mathrm{e}^{\mathrm{i}\xi^2\alpha}F_{0\eta\eta\eta}. \tag{2.45}$$

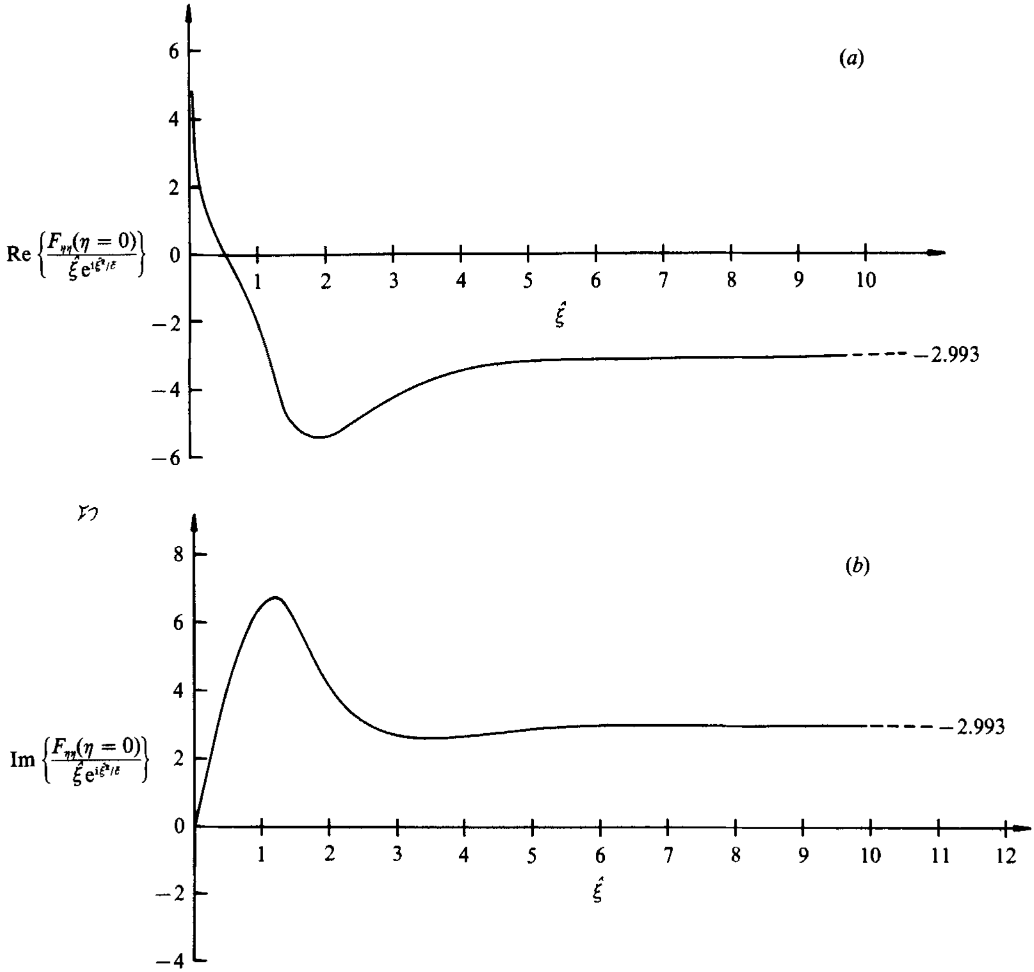

FIGURE 1. (*a*) $\mathrm{Re}\{F_{\eta\eta}(\eta = 0)/\hat{\xi}\,e^{i\hat{\xi}^2/\bar{c}}\}$ distribution; (*b*) $\mathrm{Im}\{F_{\eta\eta}(\eta = 0)/\hat{\xi}\,e^{i\hat{\xi}^2/\bar{c}}\}$ distribution $M_\infty = \sqrt{2}\,\bar{c} = 1 - 1/M_\infty$.

α may be completely eliminated from the problem by introducing the new scaled variable $\hat{\xi}$

$$\hat{\xi} = \left(\frac{c\alpha}{U_\infty}\right)^{\frac{1}{2}} \xi. \tag{2.46}$$

It is also convenient to introduce the non-dimensional wave speed

$$\bar{c} = c/U_\infty, \tag{2.47}$$

and so (2.45) becomes

$$\begin{aligned} &F_{1\eta\eta\eta} + F_0 F_{1\eta\eta} + F_1 F_{0\eta\eta} + 2i\hat{\xi}^2 F_{1\eta} - \hat{\xi} F_{0\eta} F_{1\eta\hat{\xi}} + \hat{\xi} F_{1\hat{\xi}} F_{0\eta\eta} \\ &\qquad = 2iG_0(\eta)\,\hat{\xi}^2\left(1 - \frac{1}{\bar{c}}\right) e^{i\hat{\xi}^2/\bar{c}} - \gamma M_\infty^2 (\bar{c} - 1)\, e^{i\hat{\xi}^2/\bar{c}} F_{0\eta\eta\eta}, \end{aligned} \tag{2.48}$$

subject to

$$F_1(\eta = 0) = \Phi_{1\eta}(\eta = 0) = 0, \quad F_{1\eta} \to e^{i\hat{\xi}^2/\bar{c}} \quad \text{as} \quad \eta \to \infty. \tag{2.49}$$

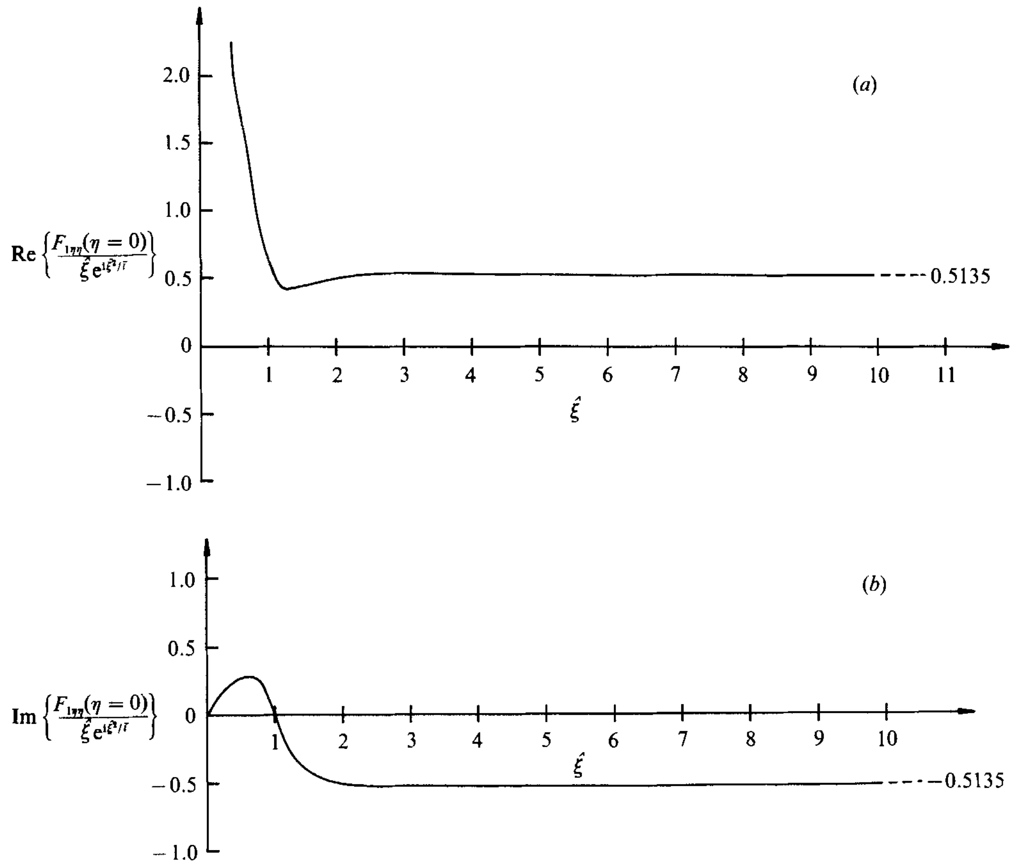

FIGURE 2(*a*, *b*). As figure 1 except $\bar{c} = 1 + 1/M_\infty$.

This equation may be viewed as the compressible counterpart of the work of Patel (1975), and also of Ackerberg & Phillips (1972) (and others cited previously) which was specifically concerned with temporal oscillations.

Initial conditions for this system were obtained by setting $\hat{\xi} = 0$ in (2.48), which yields the following (ordinary) differential system:

$$F_{1\eta\eta\eta} + F_0 F_{1\eta\eta} + F_1 F_{0\eta\eta} = -\gamma M_\infty^2 (\bar{c} - 1) F_{0\eta\eta\eta}, \tag{2.50}$$

and then solving for F_1 (using (2.49) as boundary conditions).

Equations (2.38)–(2.40) were solved numerically, using a second-order finite-difference scheme, in which the momentum and energy equations were split into three and two first-order differential equations respectively, and solved iteratively using Newton's method. $F_0(\eta)$ and $G_0(\eta)$ were then put into (2.48); this (linear) equation was solved using a similar second-order finite-difference scheme in η, Crank–Nicolson marching in $\hat{\xi}$, again by splitting the equation into a system of first-order equations (in η).

Results for $e^{-i\hat{\xi}^2/\bar{c}} F_{1\eta\eta}(\eta = 0)/\hat{\xi}$ are shown in figures 1–4 (this quantity being linked directly to the perturbation wall shear). In all cases we set $\gamma = 1.4$, $\sigma = 0.72$, $\lambda = 0$ (plane waves) and the results are believed to be correct to within the graphical tolerance of the figures. Figure 1(*a*, *b*) shows results for $M_\infty = \sqrt{2}$, $\bar{c} = 1 - 1/M_\infty$,

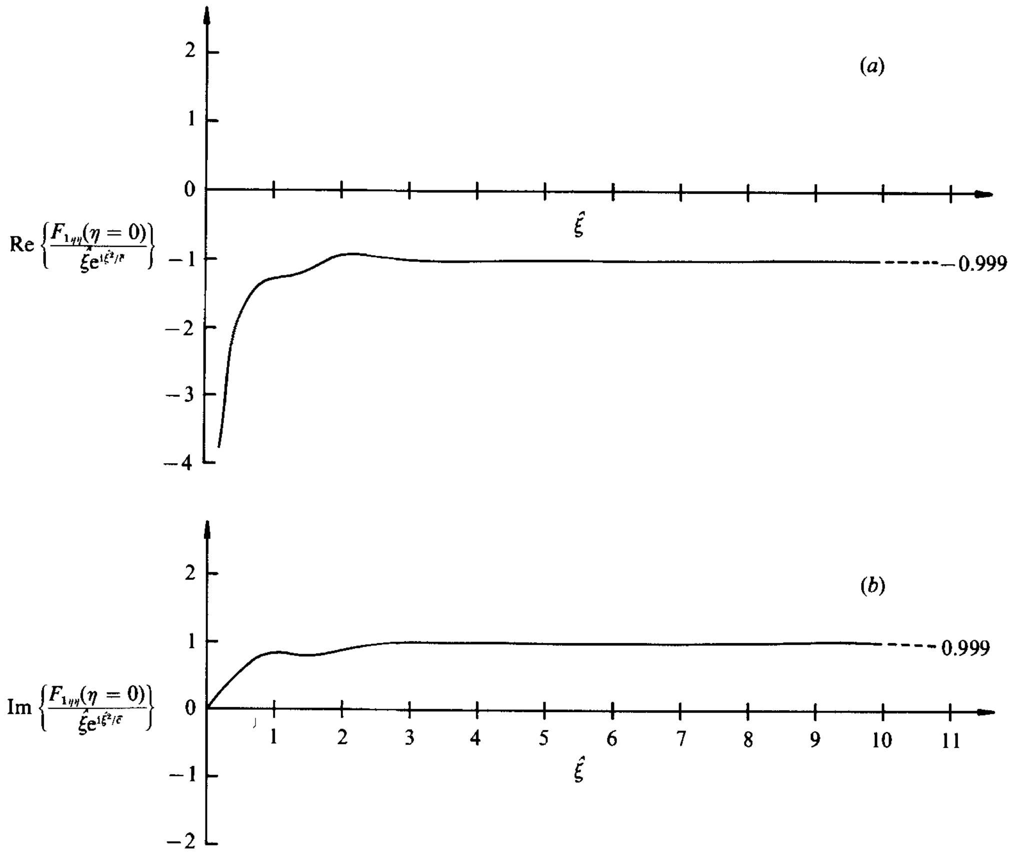

FIGURE 3(*a*, *b*). As figure 1 except $M_\infty = 5$.

figure 2(*a*, *b*) for $M_\infty = \sqrt{2}$, $c = 1+1/M_\infty$, figure 3(*a*, *b*) for $M_\infty = 5$, $\bar{c} = 1-1/M_\infty$, and figure 4(*a*, *b*) for $M_\infty = 5$, $\bar{c} = 1+1/M_\infty$.

It is readily apparent from these results that as $\xi \to \infty$ (i.e. far downstream), $e^{-i\hat{\xi}^2/\bar{c}} F_{1\eta\eta}(\eta = 0)/\hat{\xi}$ asymptotes to a constant, and it is a simple matter to confirm this analytically.

From (2.48), we have as $\hat{\xi} \to \infty$

$$2i\hat{\xi}^2 F_{1\eta} - \hat{\xi} F_{0\eta} F_{1\eta\hat{\xi}} + \hat{\xi} F_{1\hat{\xi}} F_{0\eta\eta} = 2iG_0(\eta)\,\hat{\xi}^2(1-1/\bar{c})\,e^{i\hat{\xi}^2/\bar{c}}. \tag{2.51}$$

The inclusion of the second and third terms here is most easily justified by transforming to the (non-dimensional) streamwise coordinate X, where

$$X = \hat{\xi}^2, \tag{2.52}$$

whence
$$iF_{1\eta} - F_{0\eta} F_{1\eta X} + F_{0\eta\eta} F_{1X} = iG_0(\eta)\,(1-1/\bar{c})\,e^{iX/\bar{c}}. \tag{2.53}$$

Writing
$$F_1(\eta, X) = \hat{F}_1(\eta)\,e^{iX/\bar{c}}, \tag{2.54}$$

then
$$\hat{F}_1(\eta) = (\bar{c} - F_{0\eta}) \int_0^\eta \frac{G_0(\eta)\,(\bar{c}-1)}{[\bar{c} - F_{0\eta}]^2}\,d\eta. \tag{2.55}$$

This solution generally violates the no-slip condition on $\eta = 0$, since

$$\hat{F}_{1\eta}(\eta = 0) = G_0(0)\,(1-1/\bar{c}). \tag{2.56}$$

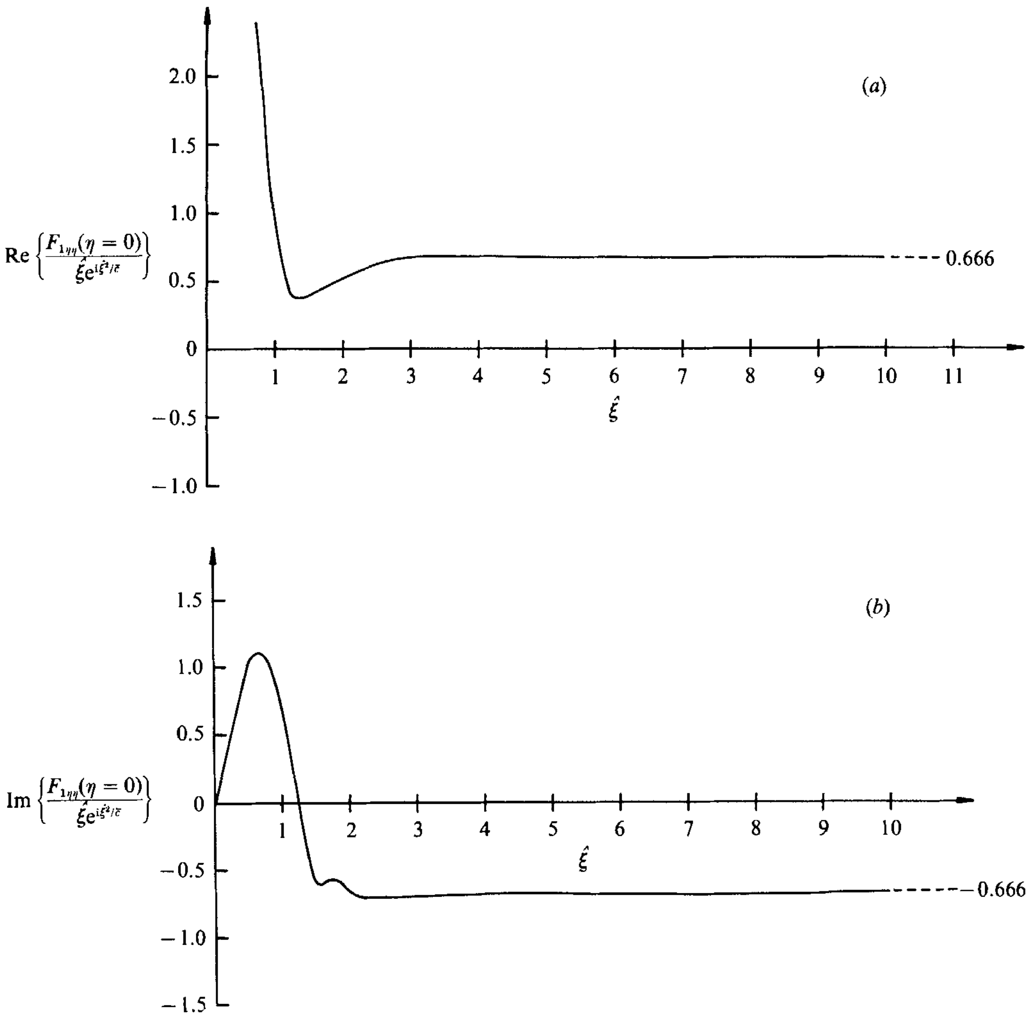

FIGURE 4 (*a*, *b*). As figure 2 except $M_\infty = 5$.

However this is easily rectified by the inclusion of a Stokes layer, where

$$Y = \eta\hat{\xi} = \eta X^{\frac{1}{3}} = O(1), \tag{2.57}$$

and

$$F_{1YYY} + 2\mathrm{i}F_{1Y} = 2\mathrm{i}G_0(0)\,(1-1/\bar{c})\,\mathrm{e}^{\mathrm{i}X/\bar{c}}. \tag{2.58}$$

Writing

$$F_1(Y,X) = \hat{f}_1(Y)\,\mathrm{e}^{\mathrm{i}X/\bar{c}}, \tag{2.59}$$

then

$$\hat{f}_{1Y}(Y) = G_0(0)\,(1-1/\bar{c})\,[1-\mathrm{e}^{-(1-\mathrm{i})\,Y}], \tag{2.60}$$

which does satisfy the no-slip condition on $Y = 0$ and also as $Y \to \infty$ matches to (2.55) as $\eta \to 0$. Equation (2.60) then gives

$$F_{1\eta\eta}(\eta = 0) = \hat{\xi}\,\mathrm{e}^{\mathrm{i}\hat{\xi}^3/\bar{c}}G_0(0)\,(1-1/\bar{c})\,(1-\mathrm{i}). \tag{2.61}$$

This is shown as an asymptote on figures 1–4, and the result is seen to be confirmed. Equation (2.55) does admit the possibility of a singularity, wherever

$$F_{0\eta} = \bar{c}, \tag{2.62}$$

and a critical layer is necessary; this is of standard form, of the type described by Lees & Lin (1946).

Unfortunately the above expansion is not uniformly valid as $\hat{\xi} \to \infty$. This is most easily seen by comparing the magnitude of the transverse pressure gradient $\partial p/\partial y = O(\rho_\infty U_\infty^2/\delta)$ with the transverse inertia terms (in particular the $\rho u v_x$ term), namely

$$\frac{\partial p/\partial y}{\rho u v_x} = O\left(\frac{\rho_\infty U_\infty^2 (\rho_\infty U_\infty/\mu_\infty x)^{\frac{1}{2}}}{\rho_\infty \alpha^2 U_\infty (\mu_\infty U_\infty x/\rho_\infty)^{\frac{1}{2}}}\right) = O\left(\frac{\rho_\infty U_\infty}{\mu_\infty x \alpha^2}\right). \tag{2.63}$$

Hence there will be a breakdown to the above solution when

$$x = O\left(\frac{\rho_\infty U_\infty}{\mu_\infty \alpha^2}\right), \tag{2.64}$$

when the transverse pressure gradient can no longer be neglected. In the following section we go on to consider the repercussions of this.

One final point is that eigensolutions analogous to those described by Lam & Rott (1960) are certainly expected to occur far downstream (although their magnitude is exponentially small, and as such they were undetectable in our numerical results); however, as noted in the previous section, unlike the subsonic case, in the supersonic planar case currently under consideration, these eigensolutions are likely to be of little consequence because of the stability of planar Tollmien–Schlichting waves (in particular those captured by triple-deck theory) in supersonic flow.

However, it has been shown by Smith (1989) and Duck (1990) that unstable, oblique viscous modes in supersonic flows may be described by triple-deck theory. Consequently, we may expect that an appropriate study, based on the ideas of Lam & Rott (1960) and Goldstein (1983) would likely lead to eigensolutions which, far downstream, would become unstable. However, this aspect is not considered here; as specified earlier in this paper we concentrate solely on planar flows.

3. The far downstream region

As noted at the end of the last section, unsteady transverse pressure fluctuations will become significant in the physics of the boundary layer when

$$x = O\left(\frac{\rho_\infty U_\infty}{\mu_\infty \alpha^2}\right). \tag{3.1}$$

This implies (and is consistent with)

$$y = O(\alpha^{-1}), \tag{3.2}$$

implying that the transverse scale (i.e. the boundary-layer thickness) is comparable with the wavelength of the imposed wave. At the same time, the streamwise lengthscale of the wave (α^{-1}) is much shorter than the local developmental length of the boundary layer (i.e. (3.1)), and so the parallel flow approximation is a rational procedure, and is adopted here.

We choose to make the following non-dimensionalizations:

$$\left.\begin{aligned} &\bar{y} = \frac{y}{\delta}, \quad \bar{\alpha} = \alpha\delta, \quad (\bar{u}, \bar{v}) = \frac{(u, v)}{U_\infty}, \quad \hat{\rho} = \frac{\rho}{\rho_\infty}, \\ &\bar{p} = \frac{p}{\rho_\infty R T_\infty}, \quad \hat{T} = \frac{T}{T_\infty}, \quad \bar{c} = \frac{c}{U_\infty}, \quad \bar{t} = \frac{U_\infty t}{\delta}. \end{aligned}\right\} \tag{3.3}$$

Here δ represents the boundary-layer thickness (given by (2.37)). We then expect the solution to develop as follows

$$\left.\begin{aligned} \bar{u} &= \bar{U}(\bar{y})+\epsilon f(\bar{y})E+\ldots, \quad \bar{v} = \epsilon\bar{\alpha}\varphi(\bar{y})E+\ldots, \\ \bar{p} &= 1+\epsilon P(\bar{y})E+\ldots, \quad \hat{T} = \bar{T}(\bar{y})+\epsilon\theta(\bar{y})E+\ldots, \\ \hat{\rho} &= \bar{\rho}(\bar{y})+\epsilon R(\bar{y})E+\ldots, \end{aligned}\right\} \tag{3.4}$$

with
$$E = \mathrm{e}^{\mathrm{i}\bar{\alpha}(x-ct)},$$

and where $\bar{U}(\bar{y})$, $\bar{\rho}(\bar{y})$ and $\bar{T}(\bar{y})$ denote the mean flow speed, density and temperature respectively. (More formally, weak, non-parallel effects could be included by introducing the slowly varying streamwise variable $x_1 = x\delta$, and allowing the perturbation terms in (3.4) to be functions also of this variable; however, these effects will not be considered in this study).

The governing equations are then found to be (Lees & Lin 1946)

$$\bar{\rho}[\mathrm{i}(\bar{U}-\bar{c})f+\bar{U}'\varphi] = -\frac{\mathrm{i}P}{\gamma M_\infty^2}, \tag{3.5}$$

$$\bar{\rho}\mathrm{i}\bar{\alpha}^2(\bar{U}-\bar{c})\varphi = -\frac{P'}{\gamma M_\infty^2}, \tag{3.6}$$

$$\mathrm{i}(\bar{U}-\bar{c})R+\bar{\rho}(\varphi'+\mathrm{i}f)+\bar{\rho}'\varphi = 0, \tag{3.7}$$

$$P = \frac{R}{\bar{\rho}}+\frac{\theta}{\bar{T}}, \tag{3.8}$$

$$\mathrm{i}(\bar{U}-\bar{c})\theta+\bar{T}'\varphi = \mathrm{i}\left(\frac{\gamma-1}{\gamma}\right)P(\bar{U}-\bar{c})\bar{T}. \tag{3.9}$$

After some simple manipulation (following Lees & Lin 1946), we arrive at the following equation for φ:

$$\frac{\mathrm{d}}{\mathrm{d}\bar{y}}\left\{\frac{(\bar{U}-\bar{c})\varphi'-\bar{U}'\varphi}{\bar{T}-M_\infty^2(\bar{U}-\bar{c})^2}\right\}-\frac{\bar{\alpha}^2(\bar{U}-\bar{c})\varphi}{\bar{T}} = 0. \tag{3.10}$$

(It is worth noting at this stage that the analogous governing equation of Mack 1984, 1987 appears inconsistent with this equation, and should be, in Mack's notation

$$D\left\{\frac{(\tilde{\alpha}\tilde{U}-\omega)D\hat{v}-\tilde{\alpha}D\tilde{U}\hat{v}}{(1-\bar{M}^2)\tilde{T}}\right\}-\frac{\tilde{\alpha}^2(\tilde{\alpha}\tilde{U}-\omega)\hat{v}}{\tilde{T}} = 0. \tag{3.11}$$

This form is entirely consistent however, with the form given by Mack 1965.)

Finally, we follow the same approach as used previously, by using the transformation

$$\frac{\partial\tilde{Y}}{\partial\bar{y}} = \bar{\rho}(\bar{y}) \tag{3.12}$$

(analogous to (2.22)) and so (3.10) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{Y}}\left\{\frac{[\bar{U}(\tilde{Y})-\bar{c}]\varphi_{\tilde{Y}}-U'(\tilde{Y})\varphi}{\bar{T}[\bar{T}-M_\infty^2(\bar{U}(\tilde{Y})-\bar{c})^2]}\right\}-\bar{\alpha}^2(\bar{U}-\bar{c})\varphi = 0, \tag{3.13}$$

where
$$U(\tilde{Y}) = F_0'(\tilde{Y}), \quad \bar{T}(\tilde{Y}) = G_0(\tilde{Y}), \tag{3.14}$$

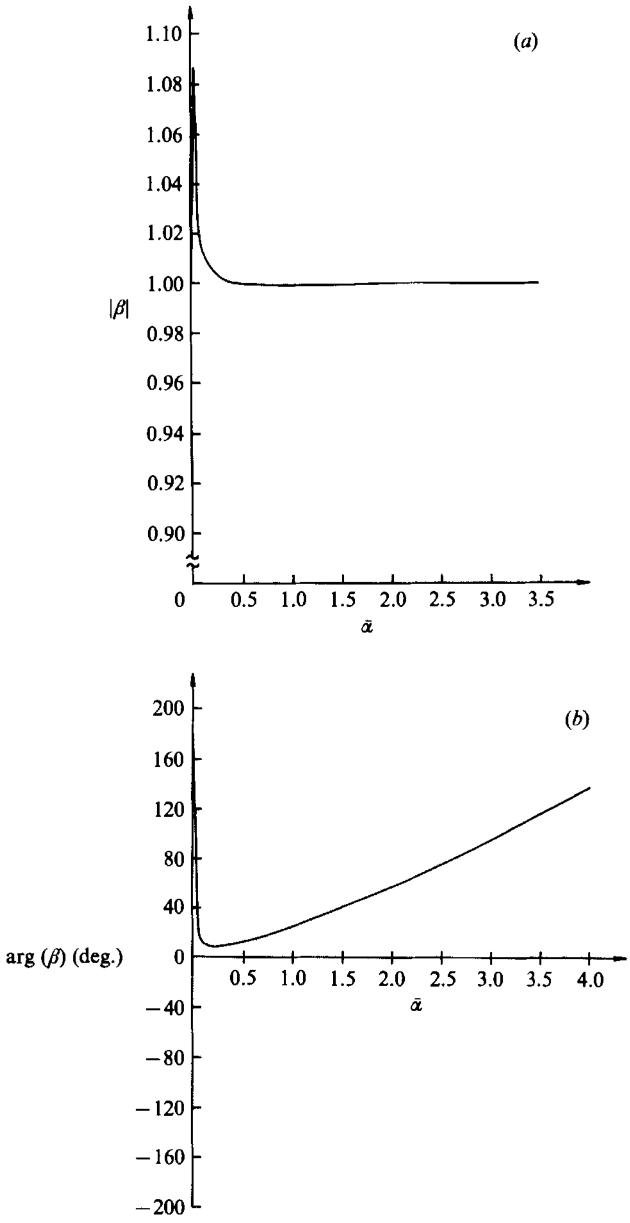

FIGURE 5. (*a*) Amplitude of β and (*b*) phase of β, for $M_\infty = \sqrt{2}$, $\lambda = 0.1$, $\bar{c} = 1-(1+\lambda^2)^{\frac{1}{2}}/M_\infty$.

with $F_0(\tilde{Y})$ defined by (2.38), and $G_0(\tilde{Y})$ by (2.39), with $\tilde{Y}$ replacing η.

The boundary condition on $\tilde{Y} = 0$ is

$$\varphi(\tilde{Y} = 0) = 0, \tag{3.15}$$

corresponding to the impermeability condition. As $\tilde{Y} \to \infty$, we require the match with the progressive-wave solution. From the previous section, this condition can be described by the superposition of an incoming and outgoing wave, if

$$\varphi \sim e^{i\bar{\alpha}\lambda\tilde{Y}} + \beta\, e^{-i\bar{\alpha}\lambda\tilde{Y}} \quad \text{as} \quad Y \to \infty. \tag{3.16}$$

β is a reflection coefficient, which is to be determined as part of the solution. The wavespeed $\bar{c}$ is given by

$$\bar{c} = 1 \pm \frac{1}{M_\infty}(1+\lambda^2)^{\frac{1}{2}} \tag{3.17}$$

on account of (2.12). The problem then may be thought of as that to determine β, given λ (generally complex), M_∞, and $\bar{\alpha}$.

Anticipating a numerical treatment of (3.13) (or its equivalent), we choose to follow a formulation suggested by Lees & Lin (1946), writing the system in terms of φ and P, namely

$$\frac{\partial \varphi}{\partial \tilde{Y}} = \frac{\mathrm{i}\bar{T}[\bar{T}-M_\infty^2(\bar{U}-\bar{c})]P}{\gamma M_\infty^2(\bar{U}-\bar{c})} + \frac{\bar{U}_{\tilde{Y}}\varphi}{\bar{U}-\bar{c}} \tag{3.18}$$

and

$$\frac{\partial P}{\partial \tilde{Y}} = -\mathrm{i}\gamma M_\infty^2\, \bar{\alpha}^2 M_\infty^2(\bar{U}-\bar{c})\,\varphi \tag{3.19}$$

(equation (3.19) arises directly from (3.6)). The advantage of using the system (3.18) and (3.19) instead of (3.10) (for example) is that (as noted by Lees & Lin 1946) in the case of real values of $\bar{c}$, such that

$$\bar{c} < 1 - 1/M_\infty \tag{3.20}$$

(classified as a supersonic disturbance), the apparent singularity of (3.10), at the value of $\tilde{Y}$ at which

$$\bar{T}(\tilde{Y}) = M_\infty^2[\bar{U}(\tilde{Y})-\bar{c}]^2 \tag{3.21}$$

(for $\bar{c} > 1-1/M_\infty$, the quantity $\bar{T}-M_\infty^2(\bar{U}-\bar{c})^2$ in our case will not have a zero), is seen by (3.18) and (3.19) to be just a singular point of the ordinary differential system, but not of the solution. Thus, from the numerical point of view the system defined by (3.18) and (3.19) has clear numerical advantages over other systems, such as (3.10). To be consistent with (3.16), we write

$$\varphi = \hat{\varphi}(\tilde{Y}) + \mathrm{e}^{\mathrm{i}\bar{\alpha}\lambda\tilde{Y}}; \tag{3.22}$$

corresponding to this we write

$$P = \hat{\pi}(\tilde{Y}) + \bar{\pi}\,\mathrm{e}^{\mathrm{i}\bar{\alpha}\lambda\tilde{Y}}, \tag{3.23}$$

where

$$\bar{\pi} = \frac{\lambda\bar{\alpha}\gamma M_\infty^2(1-\bar{c})}{[1-M_\infty^2(1-\bar{c})^2]}. \tag{3.24}$$

Substitution of (3.22) and (3.23) into (3.18) and (3.19) yields the following system for $\hat{\varphi}(\tilde{Y})$ and $\hat{\pi}(\tilde{Y})$:

$$\hat{\varphi}_{\tilde{Y}} + \mathrm{i}\lambda\bar{\alpha}\,\mathrm{e}^{\mathrm{i}\lambda\bar{\alpha}\tilde{Y}} = \frac{\mathrm{i}\bar{T}[\bar{T}-M_\infty^2(\bar{U}-\bar{c})^2]}{\gamma M_\infty^2(\bar{U}-\bar{c})}[\hat{\pi}+\bar{\pi}\,\mathrm{e}^{\mathrm{i}\bar{\alpha}\lambda\tilde{Y}}] + \frac{\bar{U}_{\tilde{Y}}}{\bar{U}-\bar{c}}[\hat{\varphi}+\mathrm{e}^{\mathrm{i}\lambda\bar{\alpha}\tilde{Y}}], \tag{3.25}$$

$$\hat{\pi}_{\tilde{Y}} = -\mathrm{i}\gamma M_\infty^2\,\bar{\alpha}^2(\bar{U}-\bar{c})\,[\hat{\varphi}+\mathrm{e}^{\mathrm{i}\bar{\alpha}\lambda\tilde{Y}}] - \mathrm{i}\bar{\alpha}\lambda\bar{\pi}\,\mathrm{e}^{\mathrm{i}\bar{\alpha}\lambda\tilde{Y}}, \tag{3.26}$$

with

$$\hat{\varphi}|_{\tilde{Y}=0} = 0, \tag{3.27}$$

and, as $\tilde{Y}\to\infty$,

$$\bar{\pi}\,\mathrm{e}^{-\mathrm{i}\bar{\alpha}\lambda\tilde{Y}}\varphi + \hat{\pi}\,\mathrm{e}^{\mathrm{i}\lambda\bar{\alpha}\tilde{Y}} \to 0. \tag{3.28}$$

The system (3.25)–(3.28) was solved using a Runge–Kutta scheme; in the case of real values of $\bar{c}$, with $|\bar{c}| < 1$, the numerical scheme may be diverted into the complex $\tilde{Y}$-plane (below the real $\tilde{Y}$-axes) at points close to the critical layer, where $\bar{U}(\tilde{Y}) = \bar{c}$ (as described by Mack 1965).

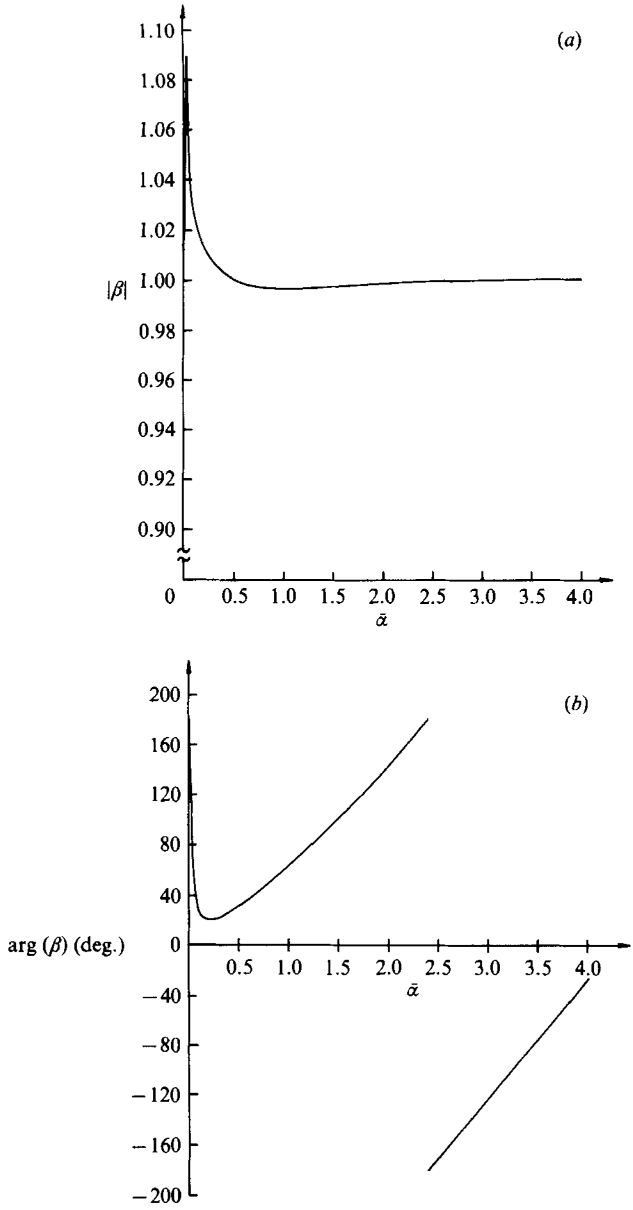

FIGURE 6. (a) Amplitude of β and (b) phase of β, for $M_\infty = \sqrt{2}$, $\lambda = 0.25$, $\bar{c} = 1-(1+\lambda^2)^{\frac{1}{2}}/M_\infty$.

This solution will in general fail to satisfy the no-slip condition; however, this is easily remedied by the presence of a Stokes layer of thickness $Y = O(1)$ (see (2.56)).

Figures 5–10 show the variation of the amplitude and phase (restricted to the range $-180°$ to $180°$) of the reflection coefficient β, with wavenumber $\bar{\alpha}$. We confine our results to real values of λ, and in all cases take the negative root of (3.17) (although there is no conceptional difficulty in extending the calculation to complex λ, or taking the positive root in (3.17); indeed, these cases are somewhat easier to calculate, since then there no longer exists a critical layer on the real $\tilde{Y}$-axis).

Results for $M_\infty = \sqrt{2}$, $\lambda = 0.1$ are shown in figure 5(a, b); for $M_\infty = \sqrt{2}$, $\lambda = 0.25$

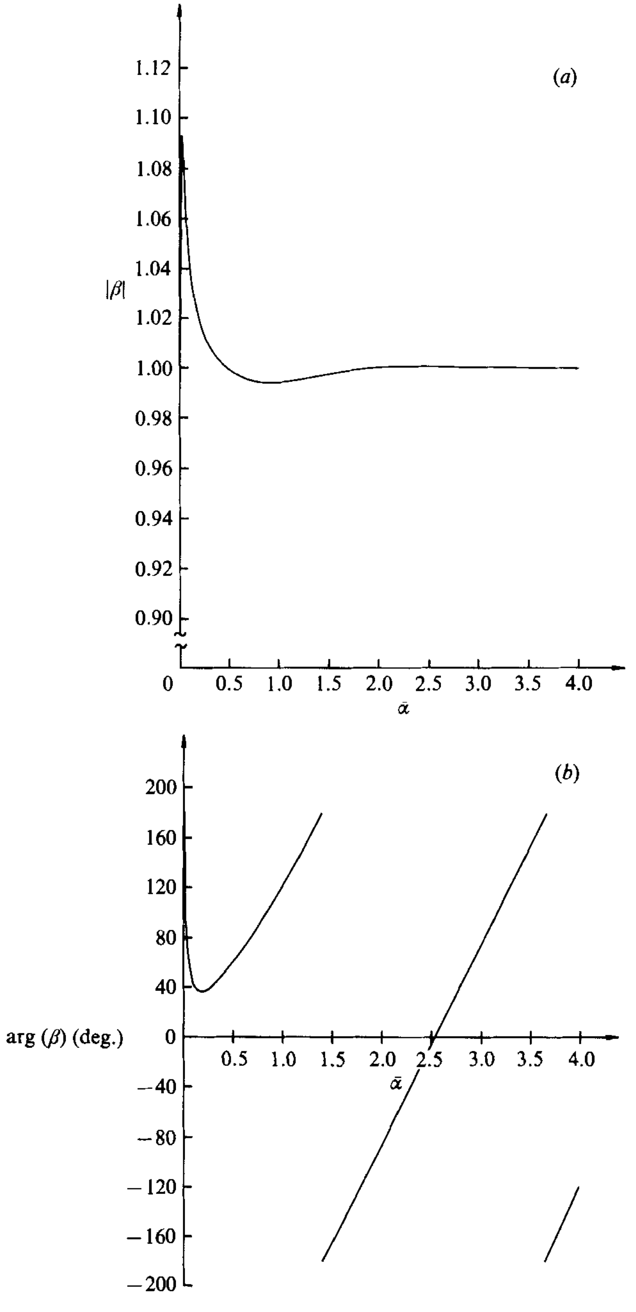

FIGURE 7. (a) Amplitude of β and (b) phase of β, for $M_\infty = \sqrt{2}$, $\lambda = 0.5$, $\bar{c} = 1-(1+\lambda^2)^{\frac{1}{2}}/M_\infty$.

in figure 6(a, b); for $M_\infty = \sqrt{2}$, $\lambda = 0.5$ in figure 7(a, b); for $M_\infty = 4.5$, $\lambda = 0.25$ in figure 8(a, b); for $M_\infty = 4.5$, $\lambda = 0.5$ in figure 9(a, b); and for $M_\infty = 4.5$, $\lambda = 1.2168$ in figure 10(a, b). All these results were obtained for $\gamma = 1.4$, $\sigma = 0.72$. Extensive numerical grid experimentation was undertaken, and the results shown are believed to be independent of numerical grid, on the scale shown. Generally a grid size of $\Delta\tilde{Y} = 0.01875$, with the grid extending out to $\tilde{Y} = 120$ was used.

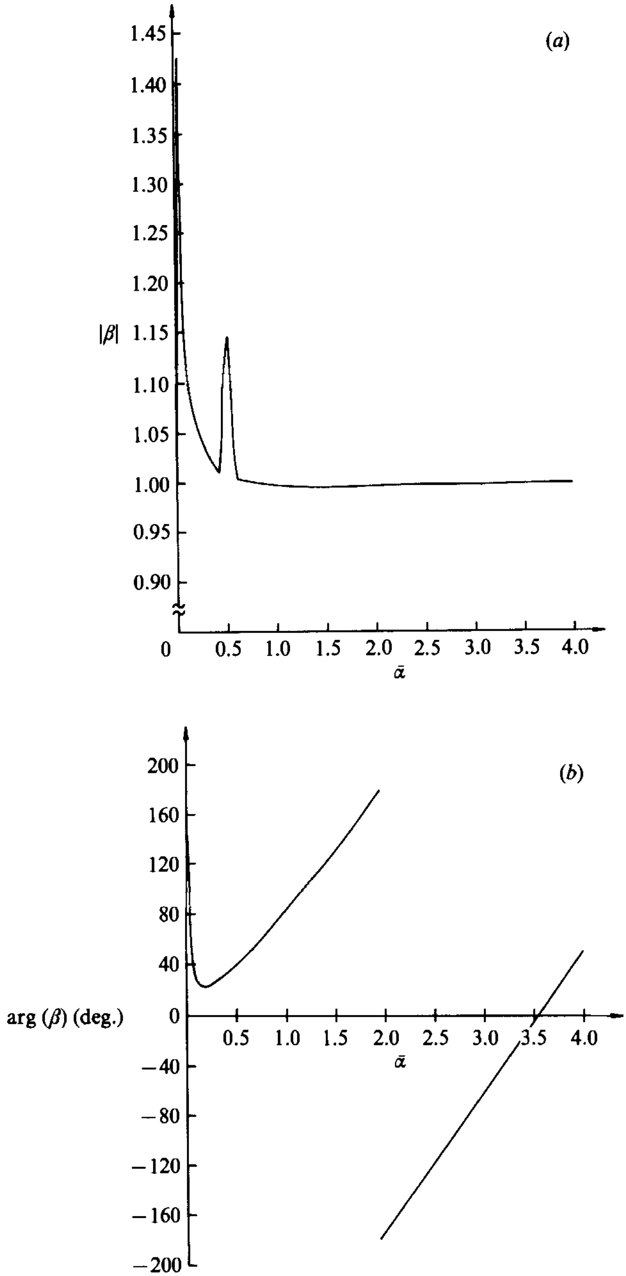

FIGURE 8. (*a*) Amplitude of β and (*b*) phase of β, for $M_\infty = 4.5$, $\lambda = 0.25$, $\bar{c} = 1-(1+\lambda^2)^{\frac{1}{2}}/M_\infty$.

One feature that is clearly visible for these computations is that $\beta \rightarrow -1$ as $\bar{\alpha} \rightarrow 0$. This (partially) indicates a correct match with the results of the previous section in the upstream limit (on account of the non-dimensionalization used to define $\bar{\alpha}$, $\bar{\alpha} \rightarrow 0$ may be interpretated as the upstream limit of these results), in particular with (2.14). This match will be studied more formally in the following section. (Further, note that the case $\bar{c} = 1 - 1/M_\infty$ and $\bar{\alpha} = 0$ corresponds to the 'sonic' eigensolutions, as studied by Lees & Lin 1946.)

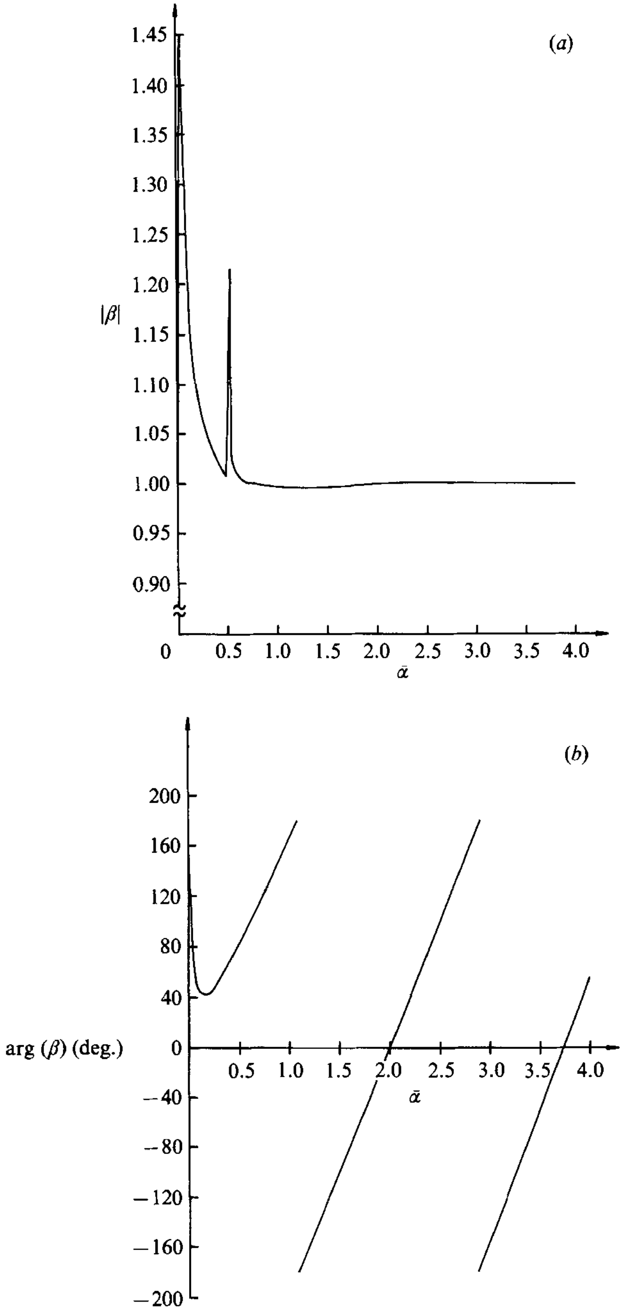

FIGURE 9. (a) Amplitude of β and (b) phase of β, for $M_\infty = 4.5$, $\lambda = 0.5$, $\bar{c} = 1-(1+\lambda^2)^{\frac{1}{2}}/M_\infty$.

A second trend observed in the results is the large variation of both the amplitude and phase of β, with $\bar{\alpha}$, as $\bar{\alpha}\to 0$; this effect appears more pronounced as $\lambda \to 0$ also. It turns out that both trends can be described by asymptotic analysis.

We anticipate that as $\bar{\alpha}\to 0$, the key scale for λ will be $O(\bar{\alpha})$ (or vice versa, of course). Formally we set

$$\lambda = \bar{\alpha}\bar{\lambda}, \quad \bar{\lambda} = O(1). \tag{3.29}$$

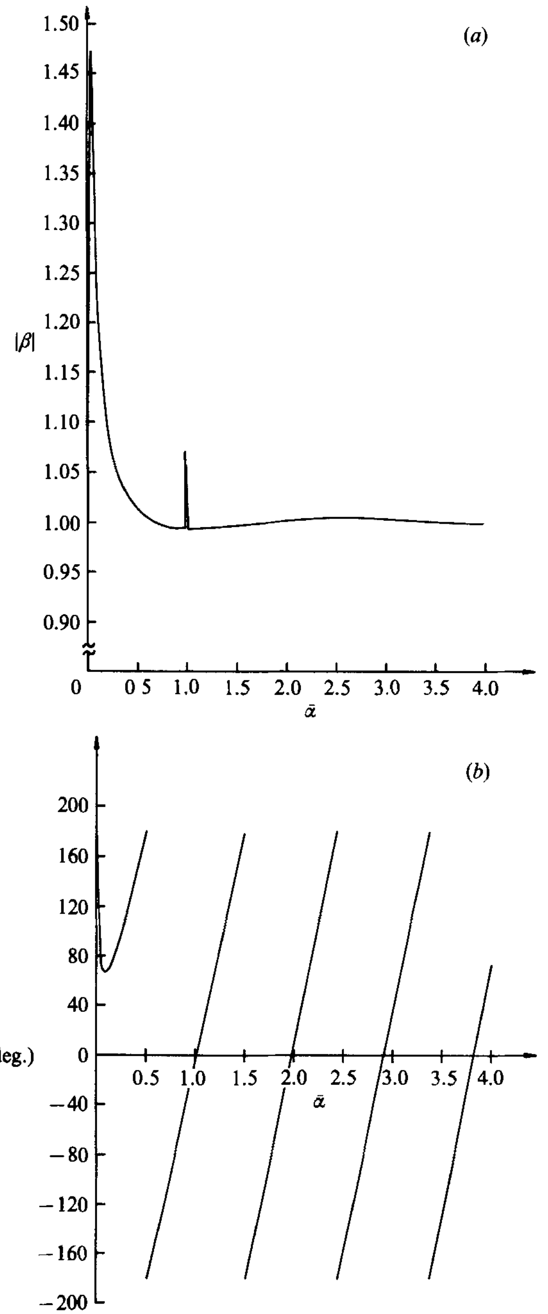

FIGURE 10. (*a*) Amplitude of β and (*b*) phase of β, for $M_\infty = 4.5$, $\lambda = 1.2168$, $\bar{c} = 1-(1+\lambda^2)^{\frac{1}{2}}/M_\infty$.

We then expand $\varphi(\tilde{Y})$ and $\bar{c}$ in the following series:

$$\varphi = \varphi_0 + \bar{\alpha}^2\varphi_1 + \dots, \tag{3.30}$$

$$\bar{c} = c_0 + \bar{\alpha}^2 c_1 + \dots, \tag{3.31}$$

where, by (3.17),

$$c_0 = 1 - 1/M_\infty, \tag{3.32}$$

$$c_1 = -\bar{\lambda}^2/2M_\infty. \tag{3.33}$$

The boundary conditions to be applied to (3.30) as $\tilde{Y} \to \infty$ are

$$\varphi_0 \sim (1+\beta), \quad \varphi_{1\tilde{Y}} \sim \mathrm{i}\bar{\lambda}(1-\beta). \tag{3.34}$$

To zeroth order in $\bar{\alpha}$, equation (3.13) yields

$$\varphi_0 = K_0(\bar{U}-c)\int_0^{\tilde{Y}} \bar{T}\frac{[\bar{T}-M_\infty^2(\bar{U}-c_0]}{(\bar{U}-c_0)^2}\,\mathrm{d}\tilde{Y} \tag{3.35}$$

(see, for example, Lees & Lin 1946), where K_0 is a constant which, if (3.35) is to match with (3.34), is given by

$$K_0 = M_\infty(1+\beta)/I, \tag{3.36}$$

where
$$I = \int_0^\infty \bar{T}\frac{[\bar{T}-M_\infty^2(\bar{U}-c_0)^2}{(\bar{U}-c_0)^2}\,\mathrm{d}\tilde{Y}. \tag{3.37}$$

The $O(\bar{\alpha}^2)$ terms in (3.13) yield the following for φ_1:

$$-2M_\infty^2(\bar{U}-c_0)\,c_1 K_0 \bar{T} + (\bar{U}-c_0)\varphi_{1\tilde{Y}} - \bar{U}^2(\tilde{Y})\varphi - c_1\varphi_{0\tilde{Y}}$$
$$= \bar{T}[\bar{T}-M_\infty^2(\bar{U}-c_0)^2]\,[K_1 + \int_0^{\tilde{Y}} (\bar{U}-c_0)\varphi_0\,\mathrm{d}\tilde{Y}], \tag{3.38}$$

where K_1 is a constant. Imposing the $\tilde{Y} \to \infty$ conditions on (3.38) yields

$$2M_\infty^2\, c_1 K_0 = \mathrm{i}\bar{\lambda}(1-\beta), \tag{3.39}$$

and hence, using (3.33), (3.36), and solving for β, we obtain

$$\beta = \frac{\mathrm{i}I + M_\infty^2\,\bar{\lambda}}{\mathrm{i}I - M_\infty^2\,\bar{\lambda}}. \tag{3.40}$$

This is the key formula determining β in this regime. Notice in particular that as $\bar{\lambda} \to 0$ (equivalently, increasing $\bar{\alpha}$, maintaining λ fixed), we obtain $\beta \to 1$, whilst as $\bar{\lambda} \to \infty$ (equivalently reducing $\bar{\alpha}$, maintaining λ fixed), we obtain $\beta \to -1$. These trends are in accord with the numerical results obtained previously. Notice that (3.40) also describes (and indeed confirms) a number of the sharp gradients observed in the numerical results. These are due to the mismatch between the limit as $\lambda \to 0$ (which by the above yields $\beta \to 1$) and the limit as $\bar{\alpha} \to 0$ (which by the above yields $\beta \to -1$). Consequently if λ is small, large gradients of β with $\bar{\alpha}$ as $\bar{\alpha} \to 0$ are inevitable.

Mack (1984) presented a distribution of reflection coefficient amplitude (only) for the particular case $M_\infty = 4.5$, $\lambda = 1.2168$, over a very narrow range of $\bar{\alpha}$; this corresponds to our set of results shown in figure 10(*a*). Over the range of $\bar{\alpha}$ shown by Mack (1984), there are a number of features similar to those found in our results (and, indeed there exists a good deal amount of qualitative agreement, although the precise details of Mack's computation are a little unclear). In particular the fundamental behaviour of $|\beta|$ as $\bar{\alpha} \to 0$ and $\bar{\alpha} \to \infty$ detailed above is also to be seen in Mack's (1984) results, together with a maximum value of $|\beta|$ which is attained at a very small value of $\bar{\alpha}$.

A further feature seen in our results at the higher Mach number is the spike in the distribution of $|\beta|$. (This feature was carefully checked for numerical accuracy in our results; however, unfortunately there is no evidence of this effect in the work of Mack 1984, although this effect occurs outside the range of $\bar{\alpha}$ considered in Mack's example.) The reason for this is unclear; indeed this spike appears to occur at larger values of $\bar{\alpha}$ as λ increases, but ultimately subsides in magnitude as λ increases. The arg (β) distribution with $\bar{\alpha}$ does not reflect this behaviour.

The third trend apparent in the numerical results described above is that, as $\bar{\alpha}$ increases, the amplitude of the reflection coefficient appears to very rapidly approach unity. We now consider the $\bar{\alpha} \to \infty$ limit asymptotically (we may equivalently interpret this as the far downstream limit of the problem). For this it turns out that instead of using (3.13) describing φ, the analogous 'P' equation is the more straightforward to analyse. This equation can be written

$$P_{\tilde{Y}\tilde{Y}} - \frac{\bar{U}(\tilde{Y})P_{\tilde{Y}}}{\bar{U}-\bar{c}} - \bar{\alpha}^2\bar{T}[\bar{T}-M_\infty^2(\bar{U}-\bar{c})^2]P = 0. \tag{3.41}$$

As $\bar{\alpha} \to \infty$, we expect this equation to take on the following approximate form:

$$P_{\tilde{Y}\tilde{Y}} - \bar{\alpha}^2\Theta P = 0, \tag{3.42}$$

where

$$\Theta = \bar{T}[\bar{T}-M_\infty^2(\bar{U}-\bar{c})^2]. \tag{3.43}$$

If we consider the regime of $\bar{c}$ considered above, we have

$$\begin{array}{lll} \text{I} & \Theta > 0 \quad \text{for} \quad 0 \leqslant \tilde{Y} < Y_c, \\ \text{II} & \Theta = 0 \quad \text{for} \quad \tilde{Y} = Y_c, \\ \text{III} & \Theta < 0 \quad \text{for} \quad \tilde{Y} > Y_c. \end{array} \tag{3.44}$$

We now consider the WKBJ solutions to the system (3.42), with $\tilde{Y} = Y_c$ being a turning point. For $0 \leqslant \tilde{Y} < Y_c$ we write

$$P_{\text{I}} = \frac{A_0}{\Theta^{\frac{1}{4}}}\left\{\exp\left[\bar{\alpha}\int_{Y_c}^{\tilde{Y}}\Theta^{\frac{1}{2}}\,\mathrm{d}\tilde{Y}\right] + \exp\left[-\bar{\alpha}\int_{Y_c}^{\tilde{Y}}\Theta^{\frac{1}{2}}\,\mathrm{d}\tilde{Y} - 2\bar{\alpha}I^*\right]\right\}, \tag{3.45}$$

where

$$I^* = \int_0^{Y_c}\Theta^{\frac{1}{2}}\,\mathrm{d}\tilde{Y}. \tag{3.46}$$

Notice that (3.45) satisfies the appropriate boundary condition on $\tilde{Y} = 0$ ($\varphi = P_{\tilde{Y}} = 0$).

Close to $\tilde{Y} = Y_c$, the approximate solution can be written in terms of Airy functions:

$$P_{\text{II}} = B_1\,\mathrm{Ai}\,(-\eta^*) + B_2\,\mathrm{Bi}\,(-\eta^*), \tag{3.47}$$

where

$$\eta^* = \bar{\alpha}^{\frac{2}{3}}[-\Theta'(Y_0)]^{\frac{1}{3}}(\tilde{Y}-Y_c) \tag{3.48}$$

(here we are assuming $\Theta'(Y_c) < 0$). Matching P_{I} and P_{II} demands

$$\begin{aligned} B_1 &= 2A_0\pi^{\frac{1}{2}}\bar{\alpha}^{\frac{1}{6}}[-\Theta'(Y_c)]^{-\frac{1}{6}}, \\ B_2 &= A_0\pi^{\frac{1}{2}}\bar{\alpha}^{\frac{1}{6}}[-\Theta'(Y_c)]^{-\frac{1}{6}}\,\mathrm{e}^{-2\bar{\alpha}I^*}. \end{aligned} \tag{3.49}$$

In the region $Y_c < \tilde{Y} < \infty$, the WKBJ solution for P can be written

$$P_{\text{III}} = P^*\left[\frac{\Theta(\infty)}{\Theta(Y)}\right]^{\frac{1}{4}}\left\{\exp\left[\mathrm{i}\bar{\alpha}\int_{Y_c}^{\tilde{Y}}(-\Theta)^{\frac{1}{2}}\,\mathrm{d}\tilde{Y}\right] + B^*\exp\left[-\mathrm{i}\bar{\alpha}\int_{Y_c}^{\tilde{Y}}(-\Theta)^{\frac{1}{2}}\,\mathrm{d}\tilde{Y}\right]\right\}. \tag{3.50}$$

Matching P_{II} and P_{III} leads to the following expressions:

$$B^* = \frac{\pi^{-\frac{1}{2}}[B_1(1+\mathrm{i}) + B_2(1-\mathrm{i})][-\Theta'(Y_c)]^{\frac{1}{6}}}{2\sqrt{2}\lambda^{\frac{1}{2}}\bar{\alpha}^{\frac{1}{6}}P^*}, \tag{3.51}$$

together with

$$A_0 = \frac{\lambda(1+\mathrm{i})P^*}{\sqrt{2}[1+\frac{1}{2}\mathrm{i}\,\mathrm{e}^{2\bar{\alpha}I^*}]}. \tag{3.52}$$

After some algebra, this yields the important result

$$B^* = \frac{2\mathrm{i}+\mathrm{e}^{-2\bar{\alpha}I^*}}{2+\mathrm{i}\,\mathrm{e}^{-2\bar{\alpha}I^*}}. \tag{3.53}$$

Utilizing these results together with (3.7) yields the following solution for φ_{III}:

$$\varphi_{\mathrm{III}} \sim \frac{-P^*[-\lambda^2\Theta(\tilde{Y})]^{\frac{1}{4}}}{\gamma\bar{\alpha}M_\infty^2(\bar{U}-\bar{c})}\left\{\exp\left[\mathrm{i}\bar{\alpha}\int_{Y_c}^{\tilde{Y}}(-\Theta)^{\frac{1}{2}}\mathrm{d}\tilde{Y}\right]-B^*\exp\left[-\mathrm{i}\bar{\alpha}\int_{Y_c}^{\tilde{Y}}(-\Theta)^{\frac{1}{2}}\mathrm{d}\tilde{Y}\right]\right\}. \tag{3.54}$$

As $\tilde{Y}\rightarrow\infty$, we expect

$$\int_{Y_c}^{\tilde{Y}}(-\Theta)^{\frac{1}{2}} = \lambda\tilde{Y}+\Lambda+o(1), \tag{3.55}$$

where (or alternatively)

$$\Lambda = \int_{Y_c}^{\infty}\{(-\Theta)^{\frac{1}{2}}-\lambda\}\,\mathrm{d}\tilde{Y}. \tag{3.56}$$

Consequently we may write (3.54) in the following form:

$$\varphi_{\mathrm{III}} \sim -\frac{P^*\lambda\exp[\mathrm{i}\bar{\alpha}\lambda\Lambda]}{\gamma\bar{\alpha}M_\infty^2(1-c)}\{\exp(\mathrm{i}\bar{\alpha}\lambda\tilde{Y})-B^*\exp(-2\mathrm{i}\bar{\alpha}\lambda\Lambda-\mathrm{i}\bar{\alpha}\lambda\tilde{Y})\}. \tag{3.57}$$

Comparing this form with (3.16) shows

$$\beta = -\left[\frac{2\mathrm{i}+\mathrm{e}^{-2\bar{\alpha}I^*}}{2+\mathrm{i}\,\mathrm{e}^{-2\bar{\alpha}I^*}}\right]\exp[-2\mathrm{i}\bar{\alpha}\lambda\Lambda]. \tag{3.58}$$

Since, in our case I^* is real and positive, and λ and Λ are both real, we have that, as $\bar{\alpha}\rightarrow\infty$,

$$\beta\rightarrow-\mathrm{i}\,\mathrm{e}^{-2\mathrm{i}\bar{\alpha}\lambda\Lambda}. \tag{3.59}$$

This clearly illustrates the unit amplitude oscillation of $\bar{\beta}$ with $\bar{\alpha}$, as $\bar{\alpha}$ increases, as found in our results. Consequently, we see that in this limit, the effect of the boundary layer is to cause a reflected wave of the same amplitude as the oncoming wave, but with a relative phase shift.

Notice also, that the result (3.58) also applies in situations in which λ is complex. In the following section we go on to show formally how the results of this section match to those of the previous section.

4. The matching of the fore and downstream regions

Let us first consider the limit of (3.11) as $\alpha\rightarrow 0$ (which as pointed out in previous section is equivalent to the upstream limit of the downstream zone). Two lengthscales for $\hat{y}$ emerge. The first, where

$$\hat{y} = \bar{\alpha}\bar{y} = O(1), \tag{4.1}$$

involves a solution to (3.11) of an incoming and outgoing wave, namely

$$\varphi = \mathrm{e}^{\mathrm{i}\lambda\hat{y}}+\beta\,\mathrm{e}^{-\mathrm{i}\lambda\hat{y}}+o(\bar{\alpha}^0). \tag{4.2}$$

We may impose the impermeability constraint on this system, which requires $\beta=-1$ and so to leading order

$$\varphi = \mathrm{e}^{\mathrm{i}\lambda\hat{y}}-\mathrm{e}^{-\mathrm{i}\lambda\hat{y}}, \tag{4.3}$$

this condition matching correctly with the outer fore-region solution (2.14); see also the analysis in the previous section corresponding to $\bar{\alpha} \to 0$.

The second important lengthscale is $\bar{y} = O(1)$ itself, where the variation of $\bar{T}$ and $\bar{U}$ must be taken into consideration. To leading order we have

$$\frac{\mathrm{d}}{\mathrm{d}\bar{y}}\left\{\frac{(\bar{U}-\bar{c})\varphi' - \bar{U}'\varphi}{\bar{T}-M_\infty^2(\bar{U}-\bar{c})^2}\right\} = 0. \tag{4.4}$$

The solution to this is

$$\varphi = \bar{C}(\bar{U}-\bar{c})\int_0^{\bar{y}} \frac{\bar{T}-M_\infty^2(\bar{U}-\bar{c})^2}{(\bar{U}-\bar{c})^2}\,\mathrm{d}\bar{y}, \tag{4.5}$$

where the impermeability condition has been imposed, and $\bar{C}$ is a constant, which is determined by matching with (4.3); this yields

$$\bar{C} = -\frac{2\bar{\alpha}\mathrm{i}(1-\bar{c})}{\lambda}. \tag{4.6}$$

This completes the details of the $\bar{\alpha} \to 0$ solution; however, for the purposes of matching with the fore region, the simplest illustration is by means of the perturbation streamwise velocity $f(\bar{y})$. By (3.5) we see that we may write

$$f(\bar{y}) = -\frac{P\bar{T}}{\gamma M_\infty^2(\bar{U}-\bar{c})} + \frac{\mathrm{i}\bar{U}'\varphi}{(\bar{U}-\bar{c})}. \tag{4.7}$$

The pressure term P is (by (3.6)) seen to be independent of $\bar{y}$ to leading order, and given by

$$P = -\mathrm{i}\bar{C}\gamma M_\infty^2, \tag{4.8}$$

and so

$$f(\bar{y}) = \frac{\mathrm{i}\bar{C}\bar{T}}{\bar{U}-\bar{c}} + \mathrm{i}\bar{U}'\bar{C}\int_0^{\bar{y}} \frac{\bar{T}}{(\bar{U}-\bar{c})^2}\,\mathrm{d}\bar{y} - \mathrm{i}\bar{U}'\bar{C}M_\infty^2\,\bar{y} \tag{4.9}$$

(if $\bar{U}-\bar{c}$, then a critical layer of the type described by Lees & Lin 1946 will be present).

We now compare the downstream limit of the fore region with (4.9). The $O(\epsilon)$ perturbation to the x-component of velocity, as $x \to \infty$ may be written

$$\tilde{u}(x,\bar{\eta}) \sim \frac{-2\hat{p}_1\,\mathrm{e}^{\mathrm{i}\bar{\alpha}(x-ct)}}{(U_\infty - c)\,\rho_\infty}\hat{F}_{1\bar{\eta}}(\bar{\eta}) + U_\infty\,\tilde{\eta}F_{0\bar{\eta}\bar{\eta}}(\bar{\eta}), \tag{4.10}$$

where $\bar{\eta}$ is defined by

$$\bar{\eta} = \tilde{Y}(\rho_\infty U_\infty/2\mu_\infty xC)^{\frac{1}{2}}, \tag{4.11}$$

whilst

$$\tilde{\eta}(\bar{\eta}) = \int_0^{\bar{\eta}} \frac{\rho(\bar{\eta})-\rho_0(\bar{\eta})}{\rho_0(\bar{\eta})}\,\mathrm{d}\bar{\eta}, \tag{4.12}$$

and $\hat{F}_1(\bar{\eta})$ is given by (2.54) (with η replaced by $\bar{\eta}$); notice that

$$\bar{\eta} = \int_0^{\bar{y}} \frac{\rho}{\rho_\infty}\,\mathrm{d}\bar{y}. \tag{4.13}$$

Equation (4.12) may be written

$$\tilde{\eta}(\bar{y}) = \int_0^{\bar{y}}\left(\frac{\tilde{p}}{\rho_\infty RT_\infty G_0(\bar{y})} - \frac{\theta_1(\bar{y})+\theta_2(\bar{y})}{T_\infty G_0^2(\bar{y})}\right)\mathrm{d}\bar{y}. \tag{4.14}$$

Here

$$\theta_1(\bar{y}) = \tilde{\eta}T_\infty G_{0\bar{y}}(\bar{y})\,G_0(\bar{y}), \tag{4.15}$$

whilst $\theta_2(\bar{y})$ is determined from the far downstream limit of (2.42) which yields

$$\theta_2(\bar{y}) = \frac{G_0(\bar{y})\,\tilde{p}}{c_p\,\rho_\infty} + \frac{T_\infty\,G_{0\bar{y}}\hat{F}_1(\bar{y})\,EC_1}{U_0-c}, \tag{4.16}$$

where we have written

$$E = \mathrm{e}^{\mathrm{i}\bar{\alpha}(x-ct)}, \tag{4.17}$$

$$U_0(\bar{y}) = U_\infty F_0'(\bar{\eta}), \tag{4.18}$$

$$C_1 = -2\hat{p}_1/[(U_\infty-c)\,\rho_\infty], \tag{4.19}$$

$$\tilde{p} = 2\hat{p}E. \tag{4.20}$$

Consequently we see that

$$\tilde{\eta}(\bar{y}) = \int_0^{\bar{y}}\left[\frac{\tilde{p}}{\rho_\infty RT_\infty\,G_0(\bar{y})} - \frac{\tilde{\eta}(\bar{y})\,G_{0\bar{y}}(\bar{y})}{G_0} - \frac{\tilde{p}}{T_\infty\,c_p\,\rho_\infty\,G_0(\bar{y})} - \frac{G_{0\bar{y}}\hat{F}_1(\bar{y})\,EC_1}{G_0(U_0-c)}\right]\mathrm{d}\bar{y}. \tag{4.21}$$

The solution of this equation is

$$\tilde{\eta}(\bar{y}) = \frac{1}{G_0(\bar{y})}\int_0^{y}\left\{\frac{\tilde{p}}{\rho_\infty RT_\infty} - \frac{\tilde{p}}{T_\infty\,c_p\,\rho_\infty} - \frac{G_{0\bar{y}}\hat{F}_1(\bar{y})\,EC_1}{U_0-c}\right\}\mathrm{d}\bar{y}. \tag{4.22}$$

We now see that as $x\to\infty$

$$\tilde{u}(x,\bar{y}) \sim C_1 E\hat{F}_{1\bar{y}}\,G_0 + U_0'(\bar{y})\,E\int_0^{\bar{y}}\left[\frac{2\hat{p}}{\gamma\rho_\infty\,T_\infty\,R} - \frac{G_{0\bar{y}}\hat{F}_1(\bar{y})\,C_1}{U_0-c}\right]\mathrm{d}\bar{y}. \tag{4.23}$$

Equation (4.23) may now be written

$$\tilde{u}(x,\bar{y}) \sim \frac{C_1 EG_0(\bar{y})\,(\bar{c}-1)}{c-U_0(\bar{y})} + \frac{2U_0'(\bar{y})\,E\hat{p}\bar{y}}{\gamma\rho_\infty\,T_\infty R} - C_1 EU_0'(\bar{y})\,(\bar{c}-1)\int_0^{\bar{y}}\frac{G_0(\bar{y})\,\mathrm{d}\bar{y}}{(c-U_0)^2} \tag{4.24}$$

$$\sim \frac{C_1 EG_0(\bar{y})\,(\bar{c}-1)}{c-U_0(\bar{y})} - C_1 M_\infty^2\,E\bar{y}U_0'(\bar{y})\,(1-\bar{c}) - C_1 EU_0'(\bar{y})\,(\bar{c}-1)\int_0^{\bar{y}}\frac{G_0(\bar{y})}{(c-U_0)^2}\mathrm{d}\bar{y}. \tag{4.25}$$

Setting

$$C_1 = \mathrm{i}U_\infty\,\bar{C}/(1-c), \tag{4.26}$$

and noting that

$$G_0(\bar{y}) = \bar{T}(\bar{y}), \tag{4.27}$$

$$U_0(\bar{y}) = \bar{U}(\bar{y}), \tag{4.28}$$

we see that (4.25) matches correctly with (4.9).

5. Discussion and conclusions

In this paper a description of the effect of a small-amplitude progressive wave on a supersonic boundary layer on a semi-infinite flat plate has been given. In this case it is possible to rule out the possibility of a receptivity problem of the same form as considered by Goldstein (1983), based on the downstream development of Lam & Rott (1960)-type eigensolutions, into unstable *planar* viscous modes (described by triple-deck theory), because these latter modes are all known to be stable in the case of supersonic flows. However, as noted in §2, in the case of oblique waves, such a description, based on the ideas of Goldstein appears possible (indeed, likely). Additionally, there appears to be no mechanism included in the present study by which initially viscous (damped), planar waves (cf. Lam & Rott 1960) may undergo a metamorphosis into (unstable) inviscid instabilities. However, it may well be that a higher-order analysis is necessary, incorporating, for example, boundary-layer growth terms (i.e. non-parallel effects) which would give an element of boundary curvature, which in turn could trigger receptivity. Indeed, receptivity is found in supersonic wind tunnels, caused by sound waves produced by turbulent tunnel-wall boundary layers.

Also included in the present study is a description of the form of the compressible Stokes layer, together with the ultimate breakdown of the (boundary layer) structure of the perturbation solution which becomes predominantly inviscid far downstream. The analysis, together with our numerical results, formally indicate a proper match between the two regimes.

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